Double limits and differentiability

Given the function: $f(x, y) = 2x^3y + y\sin\left(\frac{1}{x}\right)$

1. Compute, if they exist, at the origin: the double limit, iterated limits, and radial limit.
2. Analyze continuity and differentiability at the origin.

Solutions

1) Calculation of limits at the origin

We want to compute:

$$\lim_{(x,y)\to(0,0)} f(x,y)$$

We observe that the function $f(x,y)$ is not defined for $x=0$ due to the term $\sin\left(\frac{1}{x}\right)$. However, we can analyze the behavior of the limit as $x\to 0$ and $y\to 0$ with $x\neq 0$.

Consider that $\sin\left(\frac{1}{x}\right)$ is bounded between -1 and 1:

$$-1\leq \sin\left(\frac{1}{x}\right)\leq 1$$

Likewise, the term $2x^3y$ approaches zero as $(x,y)\to(0,0)$ because $x^3\to 0$ and $y\to 0$.

Applying the bounded-by-infinitesimal theorem:
Since $\sin\left(\frac{1}{x}\right)$ is bounded and $y\to 0$, the product $y\sin\left(\frac{1}{x}\right)$ approaches zero.
Thus, we can write:

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)}\left(2x^3y+y\sin\left(\frac{1}{x}\right)\right)=0+0=0$$

Iterated limits:
First: $\lim_{x\to 0}\left(\lim_{y\to 0} f(x,y)\right)$
Compute the inner limit:

$$\lim_{y\to 0} f(x,y)=\lim_{y\to 0}\left(2x^3y+y\sin\left(\frac{1}{x}\right)\right)=0$$

Because $y\to 0$ and the terms are proportional to y.
Now, the outer limit:

$$\lim_{x\to 0} 0=0$$

Thus:

$$\lim_{x\to 0}\left(\lim_{y\to 0} f(x,y)\right)=0$$

Second: $\lim_{y\to 0}\left(\lim_{x\to 0} f(x,y)\right)$
Compute the inner limit for $x\neq 0$:

$$\lim_{x\to 0} f(x,y)=\lim_{x\to 0}\left(2x^3y+y\sin\left(\frac{1}{x}\right)\right)$$

This limit does not exist.

Radial limit:
Define first:

$$y=mx$$

Now compute the limit:

$$\lim_{x\to 0}\left(2x^3mx+mx\sin\left(\frac{1}{x}\right)\right)$$

Using the bounded-by-infinitesimal theorem, we can assert that the limit is 0.

2) Continuity and differentiability at the origin

Continuity at the origin:

The function $f(x, y)$ is not defined for $x = 0$ due to the term $\sin\left(\frac{1}{x}\right)$. Therefore, f is not defined at $(0, 0)$, and it cannot be continuous there.

However, if we extend the definition of f at the origin by setting $f(0, 0) = 0$ (the value of the limit), we still need to verify if the function is continuous at $(0, 0)$.

Since the limit as we approach the origin is zero and $f(0, 0) = 0$, we can say that the function is continuous at $(0, 0)$ **if properly extended**.

Differentiability at the origin:

For f to be differentiable at $(0, 0)$, it must be continuous at that point and have a differential.

Compute the partial derivatives (for $x \neq 0$):

$$\frac{\partial f}{\partial x} = 6x^2y + y\cos\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)$$

$$\frac{\partial f}{\partial y} = 2x^3 + \sin\left(\frac{1}{x}\right)$$

When trying to evaluate these derivatives at $(0, 0)$, we encounter problems:

- $\frac{\partial f}{\partial x}$ is not defined for $x = 0$ due to the term $\frac{1}{x^2}$.
- $\frac{\partial f}{\partial y}$ includes the term $\sin\left(\frac{1}{x}\right)$, which is not defined for $x = 0$.

Thus, the partial derivatives at $(0, 0)$ do not exist, and f cannot be differentiable at the origin.

Conclusion:

- The function $f(x, y)$ has a limit at the origin, which is zero.
- If f is extended by defining $f(0, 0) = 0$, the function is continuous at $(0, 0)$.
- The partial derivatives at $(0, 0)$ do not exist or are not continuous, so f is not differentiable at $(0, 0)$.